



Gunns, J., & Hughes, C. (2019). The effect of repeated differentiation on L-functions. *Journal of Number Theory*, 194, 30-43.
<https://doi.org/10.1016/j.jnt.2018.07.008>

Peer reviewed version

Link to published version (if available):
[10.1016/j.jnt.2018.07.008](https://doi.org/10.1016/j.jnt.2018.07.008)

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PDF-document

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THE EFFECT OF REPEATED DIFFERENTIATION ON L -FUNCTIONS

JOS GUNNS AND CHRISTOPHER HUGHES

ABSTRACT. We show that under repeated differentiation, the zeros of the Selberg Ξ -function become more evenly spaced out, but with some scaling towards the origin. We do this by showing the high derivatives of the Ξ -function converge to the cosine function, and this is achieved by expressing a product of Gamma functions as a single Fourier transform.

1. INTRODUCTION

In 2006 Haseo Ki [5] proved a conjecture of Farmer and Rhoades [2], that differentiating the Riemann Ξ -function evens out the zero spacing. Specifically Ki showed that there exists sequences A_n and C_n with $C_n \rightarrow 0$ slowly such that

$$\lim_{n \rightarrow \infty} A_n \Xi^{(2n)}(C_n z) = \cos(z), \quad (1.1)$$

In this paper we extend Ki's result to the entire Selberg Class of L -functions, showing that there exists sequences A_n and C_n (which depend on the properties of L -function under consideration) and constants M' and θ , such that

$$\lim_{n \rightarrow \infty} A_n \Xi_F^{(2n)} \left(C_n z - \frac{M'}{\Lambda} \right) = \cos(z + \theta).$$

where Ξ_F is the Xi-function for the L -function F , an element of the Selberg Class. This result is stated more precisely in Theorem 3.1.

In [6], Selberg proposed an axiomatic definition of an L -function, now known as the Selberg Class.

Definition. A function $F(s)$ is an element of the Selberg Class if:

- (1) It has a Dirichlet series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent for $\operatorname{Re}(s) > 1$.

- (2) It is a meromorphic function such that $(s-1)^m F(s)$ is an entire function of order 1, for some integer $m \geq 0$.

Date: 1 May 2018.

2010 Mathematics Subject Classification. 11M41.

- (3) It has a functional equation of the form $\Phi(s) = \overline{\Phi(1 - \bar{s})}$, where

$$\Phi(s) = \epsilon Q^s F(s) \prod_{j=1}^k \Gamma(\lambda_j s + \mu_j)$$

with ϵ, Q, λ_j and μ_j all constants, and subject to $|\epsilon| = 1$, $Q > 0$, $\lambda_j > 0$ and $\operatorname{Re}(\mu_j) \geq 0$.

- (4) The coefficients in the Dirichlet series satisfy $a_1 = 1$ and $a_n = O(n^\delta)$ for some fixed positive δ .
- (5) It has an Euler product in the sense that

$$\log F(s) = \sum_n \frac{b_n}{n^s}$$

with $b_n = 0$ unless when $n = p^r$ for some prime p and r a positive integer, and $b_n = O(n^\theta)$ for some $\theta < 1/2$.

Kaczorowski and Perelli [4] define an Extended Selberg Class, essentially by dropping the requirement for the function to satisfy an Euler product. Our results apply equally to elements of this extended class of L -functions.

Definition. A function $F(s)$ is an element of the Extended Selberg Class if it satisfies axioms (1)–(3) above.

Remark. The degree of an L -function is 2Λ , where

$$\Lambda = \sum_{j=1}^k \lambda_j.$$

It is conjectured that the degree is always an integer. However, this is only known for L -functions of degree 2 or less [4]. More specifically, it is believed that, using the duplication formula, the gamma functions can be transformed so that $\lambda_j = 1/2$ for all j (and in such a case, the L -function has degree k).

Definition. Let F be an element of the Selberg Class, and set

$$\xi_F(s) = s^m (1-s)^m \epsilon Q^s \prod_{j=1}^k \Gamma(\lambda_j s + \mu_j) F(s).$$

Note that by assumption of F being in the Selberg Class, $\xi_F(s)$ is an entire function of order 1, with the functional equation $\xi_F(s) = \overline{\xi_F(1 - \bar{s})}$.

Definition. Set $\Xi_F(z) = \xi_F(\frac{1}{2} + iz)$.

Remark. From the functional equation $\Xi_F(z)$ is a real function for $z \in \mathbb{R}$. If the Dirichlet coefficients of F are real, then $\Xi(z)$ is an even function.

Ki proved his result for the Riemann Ξ -function by starting with the integral representation of the Gamma function to show that

$$\Xi_{\zeta}(z) = \int_{-\infty}^{\infty} \varphi(x) e^{ixz} dx,$$

where

$$\varphi(x) = 2 \sum_{n=1}^{\infty} (2n^4 \pi^2 e^{9x/2} - 3n^2 \pi e^{5x/2}) e^{-n^2 \pi e^{2x}}.$$

Note that the functional equation yields the fact that $\varphi(x) = \varphi(-x)$.

After a suitable change of variables, this yields

$$\Xi_{\zeta}(z) = 2\pi^2 \int_0^{\infty} e^{-ae^x} e^{bx} (1 + O(e^{-x})) (e^{ixz/2} + e^{-ixz/2}) dx,$$

with $a = \pi$ and $b = 9/4$. By differentiating such integrals, Ki was able to explicitly show the existence of sequences A_n and C_n such that (1.1) held. His method also holds for Hecke L -functions, since the functional equation, analogously to the Riemann Xi-function, can be written with a single Gamma function. However, the Selberg Class of L -functions generally includes a product of disparate Gamma functions, which cannot be simplified down to a single one by the multiplication formula of the Gamma function.

In sections 2 and 3, we find the Fourier transform for the analogous Ξ -function for an element of the (extended) Selberg Class of L -functions, showing it can be written as

$$\Xi_F(z) = B \int_{-\infty}^{\infty} \varphi(x) e^{i\Lambda z x} dx,$$

where $\varphi(x) = e^{-ae^x} e^{bx} (1 + O(e^{-x}))$ as $x \rightarrow \infty$, and where $\Lambda = \sum \lambda_j$.

In section 3, we start from that result to demonstrate the existence of sequences A_n and C_n such that

$$\lim_{n \rightarrow \infty} A_n \Xi_F^{(2n)} \left(C_n z - \frac{M'}{\Lambda} \right) = \cos(z + \theta)$$

where $\theta = \arg(B)$ and $M' = \sum_{j=1}^k \operatorname{Im} \mu_j$. We utilize a similar argument to that used by Ki.

The rates of convergence are considered in section 4, demonstrated by numerical examples.

Acknowledgements. This research forms part of the first author's PhD thesis from the University of York [3].

2. EXPRESSING THE Ξ -FUNCTION AS A FOURIER TRANSFORM

Theorem 2.1. *Let F be an element of the Selberg Class, with data $m, k, \varepsilon, Q, \lambda_j$, and μ_j . The Fourier transform of the Xi-function related to F is*

$$\begin{aligned}\widehat{\Xi}_F(x) &= \int_{-\infty}^{\infty} \Xi_F(z) e^{-ixz} dz \\ &= \hat{B} \exp\left(-\hat{a}e^{x/\Lambda} + \hat{b}x\right) (1 + O(e^{-x/\Lambda}))\end{aligned}$$

where

$$\hat{a} = \Lambda Q^{-1/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda}$$

and

$$\hat{b} = \frac{2m + M + \frac{1}{2}\Lambda}{\Lambda}$$

and

$$\hat{B} = (-1)^m \varepsilon Q^{-(M+2m)/\Lambda} (2\pi)^{(k+1)/2} \Lambda^{2m-1/2} \prod_{j=1}^k \lambda_j^{-\frac{1}{2} + \mu_j + \lambda_j(-M-2m)/\Lambda}$$

where

$$\Lambda = \sum_{j=1}^k \lambda_j$$

and

$$M = \sum_{j=1}^k \mu_j - \frac{1}{2}(k-1).$$

Remark. Note that Λ and M are invariant under the Gamma multiplication formulae.

Recall that

$$\begin{aligned}\Xi_F(z) &= \xi_F\left(\frac{1}{2} + iz\right) \\ &= \varepsilon Q^{1/2+iz} \left(\frac{1}{4} + z^2\right)^m F\left(\frac{1}{2} + iz\right) \prod_{j=1}^k \Gamma(i\lambda_j z + \mu_j + \tfrac{1}{2}\lambda_j)\end{aligned}$$

is an entire function. We wish to find its Fourier transform

$$\widehat{\Xi}_F(x) = \int_{-\infty}^{\infty} \Xi_F(z) e^{-ixz} dz.$$

Shifting the contour so that $F(s)$ can be represented by its Dirichlet series, swapping the order of summation and integration and shifting the contour back, we find that

$$\widehat{\Xi}_F(x) = \varepsilon Q^{1/2} \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2}} \int_{-\infty}^{\infty} \left(\frac{1}{4} + z^2\right)^m \prod_{j=1}^k \Gamma(i\lambda_j z + \mu_j + \frac{1}{2}\lambda_j) \left(\frac{ne^x}{Q}\right)^{-iz} dz. \quad (2.1)$$

Thus the Fourier transform can be found by convolutions and differentiations of the Fourier transform of the Gamma function.

Theorem 2.2 (Fourier transform of multiple gamma functions). *Let $\lambda_1, \dots, \lambda_k > 0$ and let $\alpha_1, \dots, \alpha_k$ be such that their real parts are all positive. Then for large T ,*

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\prod_{j=1}^k \Gamma(\alpha_j + i\lambda_j z) \right) e^{-iTz} dz \\ = C_k \exp \left(-\Lambda e^{T/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} + \frac{T(A - (k-1)/2)}{\Lambda} \right) (1 + O(e^{-T/\Lambda})) \end{aligned}$$

where $\Lambda = \sum_{j=1}^k \lambda_j$ and $A = \sum_{j=1}^k \alpha_j$ and

$$C_k = \frac{(2\pi)^{(k+1)/2}}{\sqrt{\Lambda}} \prod_{j=1}^k \lambda_j^{-\frac{1}{2} + \alpha_j + \lambda_j(\frac{1}{2}(k-1) - A)/\Lambda}. \quad (2.2)$$

Remark. Booker stated a similar result in the case when $\lambda_j = 1/2$ for all j , in section 5.2 of [1].

Proof. We prove this theorem by induction. The base case, when $k = 1$ says that for $\lambda > 0$ and $\text{Re}(\alpha) > 0$,

$$\int_{-\infty}^{\infty} \Gamma(i\lambda z + \alpha) e^{-iTz} dz = \frac{2\pi}{\lambda} \exp(-e^{T/\lambda} + T\alpha/\lambda). \quad (2.3)$$

This is simply the Fourier transform of one gamma function, a classical result.

With our choice of Fourier constants the convolution theorem is

$$\int_{-\infty}^{\infty} f(z)g(z)e^{-iTz} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(x)\widehat{g}(T-x)dx$$

where \widehat{f} and \widehat{g} are the Fourier transforms of f and g respectively. The Fourier transform of $k+1$ gamma functions will be the convolution of the Fourier transform of k gamma functions with the Fourier transform of one gamma function, both of

which are known by the inductive hypothesis. That is,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left(\prod_{j=1}^{k+1} \Gamma(\alpha_j + i\lambda_j z) \right) e^{-iTz} dz \\
&= \frac{C_k}{\lambda_{k+1}} \int_{-\infty}^{\infty} \exp \left(-\Lambda e^{x/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} + \frac{x(A - (k-1)/2)}{\Lambda} \right) (1 + O(e^{-x/\Lambda})) \\
&\quad \times \exp \left(-e^{(T-x)/\lambda_{k+1}} + \frac{(T-x)\alpha_{k+1}}{\lambda_{k+1}} \right) dx \quad (2.4)
\end{aligned}$$

where we have set $\Lambda = \sum_{j=1}^k \lambda_j$ and $A = \sum_{j=1}^k \alpha_j$. Later in the proof, we will also set $\Lambda' = \sum_{j=1}^{k+1} \lambda_j$ and $A' = \sum_{j=1}^{k+1} \alpha_j$.

We will asymptotically evaluate this integral. Note that the exponential in the integrand is dominated by

$$-\Lambda e^{x/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} - e^{(T-x)/\lambda_{k+1}}$$

and this has a maximum at $x = x_0$ where x_0 is such that

$$-e^{x_0/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} + \frac{1}{\lambda_{k+1}} e^{(T-x_0)/\lambda_{k+1}} = 0$$

that is

$$x_0 = \frac{T\Lambda}{\Lambda'} + \frac{\lambda_{k+1}\Lambda}{\Lambda'} \ln \left(\frac{1}{\lambda_{k+1}} \prod_{j=1}^k \lambda_j^{\lambda_j/\Lambda} \right)$$

where $\Lambda' = \Lambda + \lambda_{k+1} = \sum_{j=1}^{k+1} \lambda_j$.

Thus, expanding around $x = x_0 + \epsilon$ for small ϵ , we have (after a fair amount of straightforward algebraic simplification, and using the identity $\Lambda' = \Lambda + \lambda_{k+1}$)

$$\begin{aligned}
& -\Lambda e^{x/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} - e^{(T-x)/\lambda_{k+1}} = -e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} (\Lambda e^{\epsilon/\Lambda} + \lambda_{k+1} e^{-\epsilon/\lambda_{k+1}}) \\
&= -\Lambda' e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} \left(1 + \frac{1}{2\lambda_{k+1}\Lambda} \epsilon^2 + B_1 \epsilon^3 + O(\epsilon^4) \right)
\end{aligned}$$

where B_1 is an inconsequential constant that depends upon Λ and λ_{k+1} . (We remark that it is no surprise the coefficient of the ϵ term is zero, as this is the expansion around the maximum of the LHS).

Substituting $x = x_0 + \epsilon$ in the two other terms in the exponent of the integrand in (2.4) and letting $A' = A + \alpha_{k+1} = \sum_{j=1}^{k+1} \alpha_j$ we have

$$\begin{aligned} \frac{x(A - \frac{1}{2}(k-1))}{\Lambda} + \frac{(T-x)\alpha_{k+1}}{\lambda_{k+1}} &= \frac{T(A' - \frac{1}{2}(k-1))}{\Lambda'} \\ &+ \frac{\lambda_{k+1}(A - \frac{1}{2}(k-1)) - \alpha_{k+1}\Lambda}{\Lambda'} \ln \left(\frac{1}{\lambda_{k+1}} \prod_{j=1}^k \lambda_j^{\lambda_j/\Lambda} \right) + B_2\epsilon \end{aligned}$$

where $B_2 = \frac{A - \frac{1}{2}(k-1)}{\Lambda} - \frac{\alpha_{k+1}}{\lambda_{k+1}}$ is another inconsequential constant.

Substituting both these expansions back into (2.4) we see that the Fourier transform of the $k+1$ Gamma functions is asymptotically

$$\begin{aligned} C \exp \left(-\Lambda' e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} + \frac{T(A' - \frac{1}{2}(k-1))}{\Lambda'} \right) \\ \times \int \exp \left(-\epsilon^2 \frac{\Lambda'}{2\lambda_{k+1}\Lambda} e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} (1 + B_1\epsilon + O(\epsilon^2)) + B_2\epsilon \right) d\epsilon \end{aligned}$$

where

$$C = \frac{C_k}{\lambda_{k+1}} \left(\frac{1}{\lambda_{k+1}} \prod_{j=1}^k \lambda_j^{\lambda_j/\Lambda} \right)^{\frac{\lambda_{k+1}(A - \frac{1}{2}(k-1)) - \alpha_{k+1}\Lambda}{\Lambda'}}. \quad (2.5)$$

We utilise here the normal methods of asymptotic analysis, where the range of the ϵ integral is thought of as being small (so $O(\epsilon)$ terms are small), but $\epsilon^2 e^{T/\Lambda'}$ is large, so the Gaussian integral can be extended to the whole real line with trivially small error. To be concrete, truncate the ϵ integral to be over $[-e^{-T/3\Lambda'}, e^{-T/3\Lambda'}]$ and let $Q = \frac{\Lambda'}{2\lambda_{k+1}\Lambda} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'}$, so we have

$$\begin{aligned} \int_{-e^{-T/3\Lambda'}}^{e^{-T/3\Lambda'}} e^{-\epsilon^2 Q e^{T/\Lambda'} (1 + B_1\epsilon + O(\epsilon^2)) + B_2\epsilon} d\epsilon \\ = \int_{-e^{-T/3\Lambda'}}^{e^{-T/3\Lambda'}} e^{-\epsilon^2 Q e^{T/\Lambda'}} \left(1 - B_1 Q e^{T/\Lambda'} \epsilon^3 + B_2\epsilon + O(e^{2T/\Lambda'} \epsilon^6) \right) d\epsilon. \end{aligned}$$

We can extend the integral to be over all \mathbb{R} with a tiny error, of size $O(e^{-Qe^{T/3\Lambda'}})$. Note that due to the symmetry of the integral, the odd terms in ϵ vanish, and note that the big-O term in the integrand contributes $O(e^{-3T/2\Lambda'})$ to the integral.

Therefore, the above integral equals

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\epsilon^2 Q e^{T/\Lambda'}} \left(1 + O\left(e^{2T/\Lambda'} \epsilon^6\right)\right) d\epsilon + O\left(e^{-Q e^{T/3\Lambda'}}\right) \\ = \sqrt{\frac{\pi}{Q}} e^{-T/2\Lambda'} \left(1 + O\left(e^{-T/\Lambda'}\right)\right). \end{aligned}$$

It is easy to see the contribution to (2.4) from outside the range

$$\left[x_0 - e^{-T/3\Lambda'}, x_0 + e^{-T/3\Lambda'}\right]$$

contributes a tiny amount, dominated by the error term above, and so

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{k+1} \Gamma(\alpha_j + i\lambda_j z)\right) e^{-iTz} dz &= \sqrt{\frac{2\pi\lambda_{k+1}\Lambda}{\Lambda'}} \prod_{j=1}^{k+1} \lambda_j^{\lambda_j/(2\Lambda')} C \times \\ &\times \exp\left(-\Lambda' e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} + \frac{T(A' - \frac{1}{2}k)}{\Lambda'}\right) \left(1 + O\left(e^{-T/\Lambda'}\right)\right). \end{aligned}$$

In order to simplify the constant, recall the definitions of C given in (2.5) and C_k given in (2.2). After some rearranging, we see that

$$\begin{aligned} \sqrt{\frac{2\pi\lambda_{k+1}\Lambda}{\Lambda'}} \prod_{j=1}^{k+1} \lambda_j^{\lambda_j/(2\Lambda')} C &= \frac{(2\pi)^{(k+2)/2}}{\sqrt{\Lambda'}} \prod_{j=1}^{k+1} \lambda_j^{-1/2 + \alpha_k + \lambda_j(k/2 - A')/\Lambda'} \\ &= C_{k+1} \end{aligned}$$

which is the required form for $k+1$ Gamma functions, thus completing the proof. \square

Corollary 2.3. *Let $\lambda_1, \dots, \lambda_k > 0$ and let $\alpha_1, \dots, \alpha_k$ be such that their real parts are all positive. Then for large T ,*

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{1}{4} + z^2\right)^m \left(\prod_{j=1}^k \Gamma(\alpha_j + i\lambda_j z)\right) e^{-iTz} dz \\ = C_{k,m} \exp\left(-\Lambda e^{T/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} + \frac{T(2m + A - (k-1)/2)}{\Lambda}\right) \left(1 + O\left(e^{-T/\Lambda}\right)\right) \end{aligned}$$

where $\Lambda = \sum_{j=1}^k \lambda_j$ and $A = \sum_{j=1}^k \alpha_j$ and

$$C_{k,m} = (-1)^m (2\pi)^{(k+1)/2} \Lambda^{2m-1/2} \prod_{j=1}^k \lambda_j^{-\frac{1}{2} + \alpha_j + \lambda_j(\frac{1}{2}(k-1) - A - 2m)/\Lambda}.$$

Proof. The new term $(\frac{1}{4} + z^2)^m$ requires the first $2m$ derivatives of the RHS to be calculated. The big-O term is differentiable, and note that it dominates all the derivatives other than the $2m^{\text{th}}$ derivative. The result then follows immediately. \square

Proof of Theorem 2.1. First note that from the above Corollary, the contribution to (2.1) for the terms with $n > 1$ are exponentially smaller than the error term in $n = 1$ term, for large x . Since $a_1 = 1$ for an element of the Selberg Class, we have that for large x ,

$$\begin{aligned} \widehat{\Xi}_F(x) &= (-1)^m \varepsilon Q^{-(M+2m)/\Lambda} (2\pi)^{(k+1)/2} \Lambda^{2m-1/2} \prod_{j=1}^k \lambda_j^{-\frac{1}{2} + \mu_j - \lambda_j(M+2m)/\Lambda} \\ &\times \exp \left(-\Lambda Q^{-1/\Lambda} e^{x/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda} + (2m + M + \tfrac{1}{2}\Lambda) \frac{x}{\Lambda} \right) (1 + O(e^{-x/\Lambda})), \end{aligned}$$

where we have used the Corollary above, with $\alpha_j = \mu_j + \frac{1}{2}\lambda_j$, $T = x - \log Q$ and we set $M = \sum_{j=1}^k \mu_j - \frac{1}{2}(k-1)$. This is the theorem, with the constants \hat{B} , \hat{a} and \hat{b} given explicitly. \square

Remark. The proof made essential use of only the first three assumptions arising from $F(s)$ being an element of the Selberg class. Therefore this result holds for F an element of the Extended Selberg Class (with \hat{B} being trivially changed if $a_1 \neq 1$).

3. THE Ξ -FUNCTION UNDER REPEATED DIFFERENTIATION

Note that with our choice of constants, the inverse Fourier transform is

$$\Xi_F(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\Xi}_F(x) e^{ixz} dx.$$

Note that the μ_j , part of the data of the L -function F , could be complex. If we define

$$M' = \sum_{j=1}^k \text{Im } \mu_j,$$

and rescale z we have

$$\begin{aligned} \Xi_F \left(\frac{z - M'}{\Lambda} \right) &= \frac{\Lambda}{2\pi} \int_{-\infty}^{\infty} \widehat{\Xi}_F(x\Lambda) e^{-ixM'} e^{ixz} dx \\ &= B \int_{-\infty}^{\infty} \varphi(x) e^{ixz} dx \end{aligned}$$

where by Theorem 2.1

$$\varphi(x) = e^{-ae^x} e^{bx} (1 + O(e^{-x})), \quad (3.1)$$

with

$$a = \Lambda Q^{-1/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda}, \quad (3.2)$$

$$b = 2m + \frac{1}{2}\Lambda - \frac{1}{2}(k-1) + \sum_{j=1}^k \operatorname{Re} \mu_j \quad (3.3)$$

and $B = \hat{B}\Lambda/2\pi$. (Note that $a, b \in \mathbb{R}$ and, in the notation of Theorem 2.1, $a = \hat{a}$ and $b = \Lambda\hat{b} - iM'$).

Theorem 3.1. *Let $\Xi_F(z)$ be the Xi-function for the L-function F , an element of the Selberg Class. Let w_n be defined as the solution to*

$$aw_n e^{w_n} = bw_n + 2n$$

where a and b are given by (3.2) and (3.3) respectively. Then uniformly on compact subsets of \mathbb{C} ,

$$\lim_{n \rightarrow \infty} A_n \Xi_F^{(2n)} \left(C_n z - \frac{M'}{\Lambda} \right) = \cos(z + \arg(B)),$$

where Λ , M' , and B are given in Theorem 2.1, and the sequences A_n and C_n are given by

$$A_n = (-1)^n \exp(ae^{w_n} - bw_n) \frac{\sqrt{n}}{2|B|\Lambda^{2n}w_n^{2n+1/2}\sqrt{\pi}}$$

and

$$C_n = \frac{1}{\Lambda w_n}.$$

Remark. One can see that, for large n , the w_n defined in the theorem satisfies

$$w_n \sim \log \left(\frac{2n}{a} \right) - \log \log \left(\frac{2n}{a} \right).$$

Proof. From the functional equation for the L -function we have that

$$\Xi_F \left(\frac{z - M'}{\Lambda} \right) = \overline{\Xi_F \left(\frac{\bar{z} - M'}{\Lambda} \right)}$$

so

$$\begin{aligned} B \int_{-\infty}^{\infty} \varphi(x) e^{ixz} dx &= \overline{B} \int_{-\infty}^{\infty} \varphi(x) e^{-ixz} dx \\ &= \overline{B} \int_{-\infty}^{\infty} \varphi(-x) e^{ixz} dx, \end{aligned}$$

and since this holds for any $z \in \mathbb{C}$ we have

$$B\varphi(x) = \overline{B}\varphi(-x).$$

Therefore

$$\Xi_F \left(\frac{z - M'}{\Lambda} \right) = \int_0^\infty \varphi(x) (Be^{ixz} + \overline{B}e^{-ixz}) dx. \quad (3.4)$$

We can now consider just the integral

$$f(z) = \int_0^\infty \varphi(x) e^{ixz} dx$$

as the second integral will behave in much the same way. Differentiating this, we have that

$$f^{(2n)}(z) = (-1)^n \int_0^\infty \varphi(x) x^{2n} e^{ixz} dx.$$

Haseo Ki [5] proved that uniformly on compact subsets of \mathbb{C} ,

$$\lim_{n \rightarrow \infty} \int_0^\infty v_n \varphi(w_n x) x^{2n} e^{ixz} dx = e^{iz},$$

where $\varphi(x)$ is of the form given in (3.1), and w_n is defined such that

$$aw_n e^{w_n} = bw_n + 2n$$

and

$$v_n = \sqrt{\frac{nw_n}{\pi}} e^{ae^{w_n}} e^{-bw_n}.$$

Therefore, we have that

$$\begin{aligned} f^{(2n)}(z/w_n) &= (-1)^n \int_0^\infty \varphi(x) x^{2n} e^{ixz/w_n} dx \\ &= (-1)^n w_n^{2n+1} \int_0^\infty \varphi(w_n x) x^{2n} e^{ixz} dx \end{aligned}$$

and using Ki's work (and including the error term) we have

$$f^{(2n)}(z/w_n) = \sqrt{\frac{\pi}{nw_n}} (-1)^n e^{-ae^{w_n}} e^{bw_n} w_n^{2n+1} e^{iz} (1 + \mathcal{O}(w_n^{-2})).$$

From (3.4) we see that

$$\frac{1}{\Lambda^{2n}} \Xi_F^{(2n)} \left(\frac{z - M'}{\Lambda} \right) = B f^{(2n)}(z) + \overline{B} f^{(2n)}(-z)$$

so setting $C_n = \frac{1}{\Lambda w_n}$,

$$\begin{aligned} (-1)^n e^{ae^{w_n} - bw_n} w_n^{-2n-1} \sqrt{\frac{nw_n}{\pi}} \frac{1}{|B| \Lambda^{2n}} \Xi_F^{(2n)} \left(C_n z - \frac{M'}{\Lambda} \right) \\ = \left(\frac{B}{|B|} e^{iz} + \frac{\overline{B}}{|B|} e^{-iz} \right) (1 + \mathcal{O}(w_n^{-2})) \\ = 2 \cos(z + \arg(B)) (1 + \mathcal{O}(w_n^{-2})) \end{aligned}$$

and after taking the limit, the proof Theorem 3.1 is complete. \square

4. NUMERICAL DEMONSTRATIONS

In this section we briefly discuss how the L -function's data affects the convergence to the cosine function. Recall that the error term is $O(w_n^{-2})$ where

$$w_n \sim \log \left(\frac{2n}{a} \right),$$

with

$$a = \Lambda Q^{-1/\Lambda} \prod_{j=1}^k \lambda_j^{-\lambda_j/\Lambda}.$$

Therefore L -functions with larger conductor converge slightly more quickly, and high degree L -functions converge more slowly. This fact is clearer if one assumes that one can transform the L -function so its data has $\lambda_j = 1/2$ for all j , since then $a = kQ^{-2/k}$.

The sequence C_n effectively scales the density of the zeros of the $(2n)^{\text{th}}$ derivative. We have that

$$C_n = \frac{1}{\Lambda w_n} \rightarrow 0.$$

which means that the zeros of the unscaled $(2n)^{\text{th}}$ derivative have moved towards the origin. Compare, for example, the Riemann Xi-function before any derivatives have been taken and after 100 derivatives have been taken.

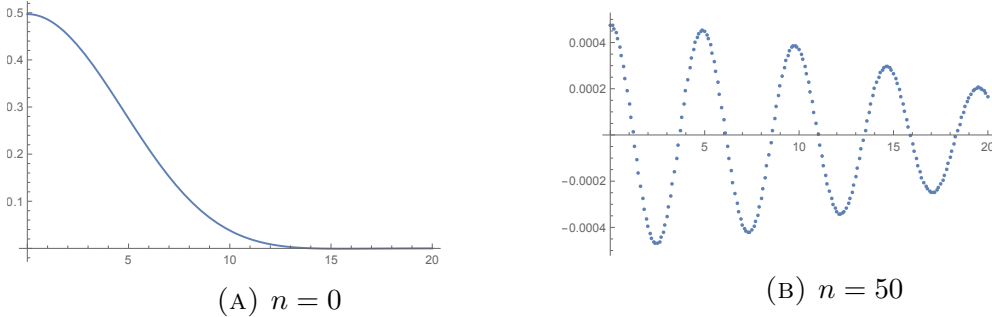


FIGURE 4.1. Plots of the Riemann Xi-function after $2n$ derivatives

These figures also demonstrate the convergence to the cosine function.

Finally, the A_n term dictates how large the derivatives of the L -functions get. From

$$A_n = \frac{\sqrt{n}(-1)^n e^{ae^{w_n}} e^{-bw_n}}{2w_n^{2n+1/2} \sqrt{\pi} |B| \Lambda^{2n}}$$

and using the defining equation for w_n , $aw_n e^{w_n} = bw_n + 2n$, we have that

$$\log |A_n| = 2n(1 - \log \Lambda - \log w_n) - ae^{w_n}(w_n - 1) + \frac{1}{2} \log n - \frac{1}{2} \log w_n + O(1)$$

and so since $w_n \sim \log(2n/a)$, as $n \rightarrow \infty$ we have that $A_n \rightarrow 0$, which means that the size of the $(2n)^{\text{th}}$ derivative gets large as n increases, although for L -functions of small degree where $\log \Lambda < 1$ the size of the derivatives can initially decrease, before eventually increasing.

REFERENCES

- [1] A.R. Booker, *Numerical tests of modularity*, J. Ramanujan Math. Soc. **20** (2005), 283–339.
- [2] D. Farmer and R. Rhoades, Differentiation evens out zero spacings, *Trans. Amer. Math. Soc.* **357** 3789–3811 (2005)
- [3] J. Gunns, Differentiating L -functions, PhD dissertation, University of York (2017)
- [4] J. Kaczorowski and A. Perelli, On the structure of the Selberg class, VII: $1 < d < 2$, *Ann. of Math. (2)* **173** 1397–1441 (2011).
- [5] H. Ki, The Riemann Ξ -function under repeated differentiation *J. Number Theory* **120** 120–131 (2006)
- [6] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, in *Proceedings of the Amalfi Conference on Number Theory 1989* 367–385 (1992)

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, BRISTOL, BS8 1TW

E-mail address: `jos.gunns@bristol.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, YO10 5DD, UNITED KINGDOM

E-mail address: `christopher.hughes@york.ac.uk`